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PUFL, An "Almost-Lagrangian" Gasdynamic Calculation for Pipe Flows with Mass Entrainment[1]

BARBARA K. CROWLEY

Lawrence Radiation Laboratory, University of California,

Livermore, California 94550

ABSTRACT

An "almost-Lagrangian" formulation in quasi-one-dimensional geometry for the conservation equations of gasdynamics is presented that considers zones with mass sources and sinks. The differential equations are first derived by an integral approach. Then, the corresponding finite-difference equations and the calculational scheme used in the PUFL code for advancing the difference equations is given. By considering uniform flows, analytical solutions for the PUFL equations with both mass sources and sinks are obtained. Stability of the "almost-Lagrangian" equations is empirically discussed and tested by comparison of PUFL calculations with the analytical solutions. It is found that when the fraction of mass in a zone changes by no more than $\sim 1\%$ in any one calculational cycle, excellent agreement with the analytical solution is obtained. By using an additional time-step control which limits the amount of mass change per cycle, the "almost-Lagrangian" formulation appears to be feasible.

INTRODUCTION

The finite-difference equations of one-dimensional, hydrodynamic calculations are frequently cast in the Lagrangian rather than the Eulerian form for several reasons. Among these reasons are: (1) mass is explicitly conserved, (2) contact surfaces are preserved, and (3) for the same order of accuracy the differential equations are simpler [1], [2]. Addition of mass to a bounded flow, as in the ablation problem, can be handled by an Eulerian formulation [3]. However, the Lagrangian formulation leads to more accurate results with less calculation, particularly in problems with large initial discontinuities [4]. The traditional Lagrangian approach cannot be used directly in problems with mass addition because zones of constant mass are assumed in the formulation. An "almost-Lagrangian" formulation that allows zones to have mass sources is, therefore, investigated.

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consideration of flow in converging–diverging nozzles and expanding pipes. Assumed in the derivations is the linearity of the sides of the element. The volume of the element, denoted by V, and the total surface area, S_T, are calculated as

$$V = \frac{\pi \Delta x}{3} [R_1^2 + R_1 R_2 + R_2^2] \text{ and} \tag{1a}$$

$$S_T = S + A_1 + A_2 = \pi(R_1 + R_2)[(\Delta x)^2 + (R_1 - R_2)^2]^{1/2} + \pi R_1^2 + \pi R_2^2, \tag{1b}$$

where S is the surface area of the zone touching the pipe wall; A_1 and A_2 are the cross-sectional areas of the surfaces of the zone normal to the flow in the pipe; R_1 and R_2 are the radii of the pipe at the locations indicated in Fig. 1; and Δx is the zone dimension measured along the center line of the pipe.

On the "wrap-around" surface, S, may be sources (or sinks) of mass. If $\dot{m}$ is the mass flux (mass/area-time) homogeneously entering the element through S, the mass entering the element in the time period Δt is $\dot{m} S \Delta t$. Associated with the mass entering the element is a velocity u_w and an internal energy per unit mass e_w. Mass fluxes associated with the surface S are encountered in ablation and boundary-layer problems. The following derivations can be applied to a mass flux between adjacent elements if $\dot{m}S$ is replaced by $\dot{m}A$ and if the sources and the sinks balance on common boundaries. Mass transfer between adjacent elements is a situation encountered, for instance, in turbulent flows where mass diffusion is considered.

The following derivations assume that all variables associated with an element, such as density, mass, pressure, and specific internal energy, are homogeneous throughout the element.

Continuity or Conservation of Mass

The time rate of change of mass in a zone may be written as

$$(Dm/Dt) = \dot{m}S, \tag{2}$$

using for the mass of the zone, $m = \rho V$, where ρ is density, we obtain

$$\frac{Dm}{Dt} = \frac{D}{Dt}\rho V = \rho \frac{DV}{Dt} + V \frac{D\rho}{Dt} = \dot{m}S. \tag{3}$$

After rearranging terms, this becomes

$$\frac{D\rho}{Dt} = \frac{1}{V}\left\{-\rho \frac{DV}{Dt} + \dot{m}S\right\} \tag{4}$$

Conservation of Momentum

The equation for the conservation of momentum may be written as

$$(Dm\mathbf{u})/Dt = \mathbf{F} + \text{sources of momentum per unit time}, \tag{5}$$

where **F** is composed of (1) body forces acting *on* the element (PUFL presently assumes that these are negligible), and (2) surface forces acting *on* the element, which are

$$\int_{S_T} \hat{n}\boldsymbol{\sigma}\, dS_T = V \operatorname{div} \boldsymbol{\sigma}, \tag{6}$$

where $\hat{n}$ is unit outward normal vector, and $\boldsymbol{\sigma}$ is the stress tensor. Eq. (5) may be rewritten

$$[m(D\mathbf{u}/Dt)] + [\mathbf{u}(Dm/Dt)] = V \operatorname{div} \sigma + \text{sources of momentum per unit time.} \tag{7}$$

Now,

$$\begin{aligned} \mathbf{u} &= u \text{ (for one-dimensional flow)} \\ V \operatorname{div} \boldsymbol{\sigma} &= -V(\partial p/\partial x) - \tau_w S \text{ (from Appendix A)} \\ Dm/Dt &= \dot{m}S \text{ (from continuity)} \\ \dot{m}Su_w &= \text{momentum per unit time from source,} \end{aligned}$$

where p is the pressure in the zone, τ_w is the shearing stress at the wall, and u_w is the velocity of the mass entering a zone.

Substituting these relationships into the momentum equation gives

$$\frac{Du}{Dt} = \frac{1}{m}\left\{\dot{m}S(u_w - u) - V\frac{\partial p}{\partial x} - \tau_w S\right\}. \tag{8}$$

Conservation of Total Energy

The total energy of an element may be modified by the following:

I. The rate of work done *on* an element by surface stresses, which is

$$\int_{S_T} \hat{n}\boldsymbol{\sigma} \cdot \mathbf{u}\, dS_T .$$

II. Sources of internal and/or kinetic energy carried by mass entering an element.

III. The rate of energy supplied *to*, or generated within, an element independent of mass sources, which is $\dot{H}$.

IV. The rate of work done *on* the system by body forces, which is $\mathbf{F} \cdot \mathbf{u}$.

From existing Lagrangian calculations in both one and two dimensions [1], it is known that various forms for the modification of energy by methods III and IV are possible in Lagrangian calculations. Because heat and mass transfer are frequently closely related, $\dot{H}$ is included in the following discussions. One example

of a negative heat flux term and an associated mass flux, which can be used to simulate the turbulent ablation process, is described in Appendix C. Since body forces are usually negligible in gasdynamic problems, they are neglected in the following.

Conservation of total energy is therefore expressed as

$$\frac{D}{Dt}(mu^2/2 + me) = \int_{S_T} \hat{n}\boldsymbol{\sigma} \cdot \mathbf{u}\, dS_T + \begin{matrix}\text{sources of energy associated} \\ \text{with mass addition}\end{matrix} + \dot{H}, \tag{9}$$

where m is the mass, e is the specific internal energy of the material in a zone, and $\dot{H}$ is the rate of heat supplied to an element. Eq. (9) may be modified as follows. From Appendix B, the rate of work done on an element by surface forces in one dimension may be written as

$$\int_{S_T} \hat{n}\boldsymbol{\sigma} \cdot \mathbf{u}\, dS_T = V\mathbf{u} \cdot \operatorname{div} \boldsymbol{\sigma} - p\frac{dV}{dt} + \tau_w S \mid u \mid \tag{9a}$$

where $\mid u \mid$ is the absolute value of the velocity.

The rate of energy addition associated with mass addition is

$$\dot{m}S(u_w^2/2 + e_w), \tag{9b}$$

where e_w is the specific internal energy of the mass entering a zone.

Differentiating the total energy gives

$$\frac{D}{Dt}(mu^2/2 + me) = mu\frac{Du}{Dt} + m\frac{De}{Dt} + (e + u^2/2)\frac{Dm}{Dt}. \tag{9c}$$

Substituting these three equations in the equation for conservation of total energy, one obtains an equation for internal energy:

$$\begin{aligned} m\frac{De}{Dt} = {} & -mu\frac{Du}{Dt} - u^2/2\frac{Dm}{Dt} - e\frac{Dm}{Dt} + V\mathbf{u} \cdot \operatorname{div} \boldsymbol{\sigma} \\ & - p\frac{DV}{Dt} + \tau_w S \mid u \mid + \dot{m}Su_w^2/2 + \dot{m}Se_w + \dot{H}. \end{aligned} \tag{10}$$

From continuity and momentum, respectively, the following relationships were found:

$$(Dm/Dt) = \dot{m}S \tag{10a}$$

$$-mu(Du/Dt) = -V\mathbf{u} \cdot \operatorname{div} \boldsymbol{\sigma} - u\dot{m}S(u_w - u). \tag{10b}$$

Substitution of these relationships in the above equation for internal energy and dividing by the mass gives

$$\frac{De}{Dt} = \frac{1}{m}\left\{-V\mathbf{u}\cdot\operatorname{div}\boldsymbol{\sigma} - u\dot{m}S(u_w - u) - \dot{m}Su^2/2 - e\dot{m}S + V\mathbf{u}\cdot\operatorname{div}\boldsymbol{\sigma}\right.$$
$$\left. - p\frac{DV}{Dt} + \tau_w S\,|\,u\,| + \dot{m}Su_w^2/2 + \dot{m}Se_w + \dot{H}\right\}. \tag{11}$$

Combining terms, one obtains

$$\frac{De}{Dt} = \frac{1}{m}\left\{\dot{m}S[(u - u_w)^2/2 + (e_w - e)] - p\frac{DV}{Dt} + \tau_w S\,|\,u\,| + \dot{H}\right\}. \tag{12}$$

II. The Finite-Difference Equations and the Calculational Procedure

The differential equations derived in Section I are now expressed in finite difference form. The pressure is modified by the addition of a dissipative term, Q, which greatly simplifies the numerical solution of hydrodynamic shock problems [6]; Q is negligibly small except in the neighborhood of the shock.

Quantities associated with the mass of the jth element (i.e., zone) are centered at $j + \frac{1}{2}$ while the zone boundaries, radii, velocities, and wall stresses are centered at j and $j + 1$. This centering is indicated in Fig. 2.

The centering of time steps is shown in Fig. 3, where it can be seen that

$$\Delta t^{n+1} = \frac{1}{2}(\Delta t^{n+1/2} + \Delta t^{n+3/2}) \tag{13}$$

Velocities are centered in time at $n - \frac{1}{2}$ and $n + \frac{1}{2}$. Other physical variables are centered at n and $n + 1$. The variables are centered in time and space in a manner customary for traditional Lagrangian difference systems of the type shown by DeBar to conserve total energy [7]. Omission of the mass and heat sources and the frictional terms from the conservation equations given below results in the traditional Lagrangian difference system.

The finite-difference equation for continuity is

$$\rho_{j+1/2}^{n+1} = \rho_{j+1/2}^{n}\frac{V_{j+1/2}^{n}}{V_{j+1/2}^{n+1}} + \Delta t^{n+1/2}\frac{\dot{m}_{j+1/2}^{n}S_{j+1/2}^{n+1/2}}{V_{j+1/2}^{n+1/2}}. \tag{14}$$

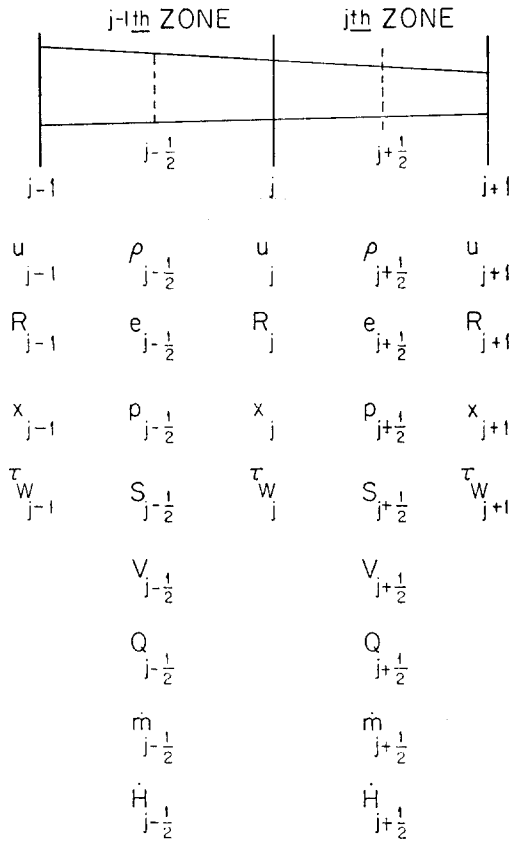

FIG. 2. Centering of calculational variables.

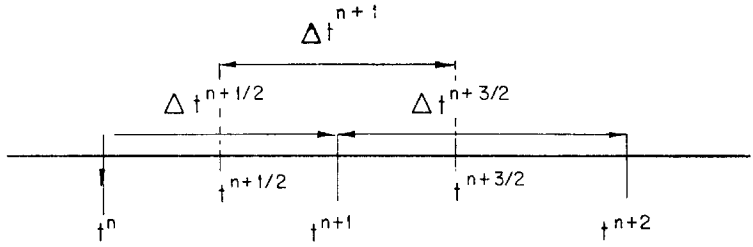

FIG. 3. Schematic relationship of time steps.

That for momentum is

$$u_j^{n+1/2} = u_j^{n-1/2} - \left\{\frac{2\Delta t^n}{m_{j+1/2}^n + m_{j-1/2}^n}\right\}\left\{\left(\frac{V_{j+1/2}^n + V_{j-1/2}^n}{x_{j+1}^n - x_{j-1}^n}\right)(\bar{P}_{j+1/2}^n - \bar{P}_{j-1/2}^n)\right.$$
$$+ (u_j^{n-1/2} - u_w)\left(\frac{\dot{m}_{j+1/2}^n S_{j+1/2}^n + \dot{m}_{j-1/2}^n S_{j-1/2}^n}{2}\right)$$
$$\left. + \tau_{w_j}^{n-1/2}\left(\frac{S_{j+1/2}^n + S_{j-1/2}^n}{2}\right)\right\}, \tag{15}$$

where

$\bar{P}_{j+1/2}^n = p_{j+1/2}^n + Q_{j+1/2}^{n-1/2}$, and $Q_{j+1/2}^{n-1/2}$ is the dissipative term mentioned above that simplifies the numerical integration. The equation for energy is

$$e_{j+1/2}^{n+1} = e_{j+1/2}^n + \frac{\Delta t^{n+1/2}\dot{m}_{j+1/2}^n S_{j+1/2}^{n+1/2}}{m_{j+1/2}^{n+1/2}}\left[\frac{(u_{j+1/2}^{n+1/2} - u_w)^2}{2} - e_{j+1/2}^{n+1/2} + e_w\right]$$
$$- \left(\frac{\bar{P}_{j+1/2}^n \Delta t^n + \bar{P}_{j+1/2}^n \Delta t^{n+1}}{2\Delta t^{n+1/2}}\right)\left(\frac{V_{j+1/2}^{n+1} - V_{j+1/2}^n}{m_{j+1/2}^{n+1/2}}\right)$$
$$+ \frac{\Delta t^{n+1/2}}{2}(\tau_{w_j}^{n+1/2} + \tau_{w_{j+1}}^{n+1/2})\frac{S_{j+1/2}^{n+1/2} u_{j+1/2}^{n+1/2}}{m_{j+1/2}^{n+1/2}} + \frac{\Delta t^{n+1/2} H_{j+1/2}^{n+1/2}}{m_{j+1/2}^{n+1/2}}. \tag{16}$$

As the equations show, e_w and u_w are effective only in zones where $\dot{m}_{j+1/2}^n \neq 0$. They are shown in the above equations as constants. Through the use of additional equations or tables, they can be treated as variables.

The calculation proceeds schematically as follows for a general cycle, n.

1. Cycle n begins.

2. Solve the momentum equation to obtain $u_j^{n+1/2}$.

3. $x_j^{n+1} = x_j^n + u_j^{n+1/2}\Delta t^{n+1/2}$,

$$x_j^{n+1/2} = \frac{1}{2}(x_j^n + x_j^{n+1}).$$

4. Determine R_j^{n+1} from whatever prescription is available. Presently radius profiles with respect to time and position are used.

5. $S_{j+1/2}^{n+1} = \pi(R_j^{n+1} + R_{j+1}^{n+1})[(x_{j+1}^{n+1} - x_j^{n+1})^2 + (R_j^{n+1} - R_{j+1}^{n+1})^2]^{1/2}$,

$$V_{j+1/2}^{n+1} = \frac{\pi(x_{j+1}^{n+1} - x_j^{n+1})}{3}\{(R_j^{n+1})^2 + R_j^{n+1}R_{j+1}^{n+1} + (R_{j+1}^{n+1})^2\},$$

$$S_{j+1/2}^{n+1/2} = \frac{1}{2}(S_{j+1/2}^{n} + S_{j+1/2}^{n+1}),$$

$$V_{j+1/2}^{n+1/2} = \frac{1}{2}(V_{j+1/2}^{n} + V_{j+1/2}^{n+1}).$$

6. Compute $\dot{m}_{j+1/2}^{n+1}$ and $\dot{H}_{j+1/2}^{n+1}$. A variety of prescriptions may be used to describe $\dot{m}_{j+1/2}^{n+1}$. The type presently used in PUFL is described in Appendix C. It relates $\dot{m}_{j+1/2}^{n+1}$ to $q_{j+1/2}^{n+1}$, a calculated heat flux out of the zone, and $\dot{H}_{j+1/2}^{n+1} = -q_{j+1/2}^{n+1}S_{j+1/2}^{n+1}$. Since zone radii may vary with time, a prescription of ablation that connects mass flux, heat flux, and radius of the zone is possible in PUFL with only minor additions.

7. Solve the continuity equation to obtain $\rho_{j+1/2}^{n+1}$.

8. $$\Delta u_{j+1/2}^{n+1/2} = \begin{cases} u_{j+1}^{n+1/2} - u_j^{n+1/2}, & \text{if } \begin{cases} u_{j+1}^{n+1/2} - u_j^{n+1/2} < 0, \\ u_{j+1}^{n+1/2} - u_j^{n+1/2} \geqslant 0 \\ \text{if the zone is expanding radially;} \end{cases} \\ 0, & \\ 0 & \end{cases}$$

$$Q_{j+1/2}^{n+1/2} = \frac{1}{2}(\rho_{j+1/2}^{n+1} + \rho_{j+1/2}^{n})\{C_0^2(\Delta u_{j+1/2}^{n+1/2})^2 - C_1 C \Delta u_{j+1/2}^{n+1/2}\},$$

where C_0^2 and C_1 are input constants (typical values are $C_0^2 \sim 3$, $C_1 \sim 1$), and C is the local sound speed. The first term with $(\Delta u)^2$ is the traditional artificial viscosity which is large at the shock front. The term that is linear in Δu provides damping for spurious oscillations in the sonic regions and is usually unnecessary.

9. $$\rho_{j+1/2}^{n+1/2} = \frac{1}{2}(\rho_{j+1/2}^{n} + \rho_{j+1/2}^{n+1}),$$

$$u_{j+1/2}^{n+1/2} = \frac{1}{2}(u_j^{n+1/2} + u_{j+1}^{n+1/2}),$$

$$m_{j+1/2}^{n+1/2} = \frac{1}{2}(m_{j+1/2}^{n} + \rho_{j+1/2}^{n+1}V_{j+1/2}^{n+1}),$$

$$R_j^{n+1/2} = \frac{1}{2}(R_j^{n} + R_j^{n+1}),$$

10. $$\tau_{w_j}^{n+1/2} = \frac{1}{2}C_f\frac{1}{2}(\rho_{j+1/2}^{n+1/2} + \rho_{j-1/2}^{n+1/2})(u_j^{n+1/2})^2,$$

where C_f is the dimensionless friction coefficient for tubes and may either require an auxiliary calculation or be treated as an input constant.

11. Solve the energy equation to obtain

$$\tilde{e}^{n+1}_{j+1/2}\text{, using } p^{n}_{j+1/2} + Q^{n-1/2}_{j+1/2} \text{ for } \bar{P}^{n+1}_{j+1/2}\text{,} \qquad \text{and} \qquad e^{n}_{j+1/2} \text{ for } e^{n+1/2}_{j+1/2}\text{.}$$

The $(\sim)^{n+1}$ quantities are a first approximation to their values at $n + 1$.

12. Calculate $\tilde{p}^{n+1}_{j+1/2}$ from an equation of state using $\tilde{e}^{n+1}_{j+1/2}$ and $\rho^{n+1}_{j+1/2}$.

13.
$$\bar{P}^{n+1}_{j+1/2} = \tilde{p}^{n+1}_{j+1/2} + Q^{n-1/2}_{j+1/2}$$

$$e^{n+1/2}_{j+1/2} = \frac{1}{2}(e^{n}_{j+1/2} + \tilde{e}^{n+1}_{j+1/2}).$$

14. Solve the energy equation to obtain $e^{n+1}_{j+1/2}$ using $\bar{P}^{n+1}_{j+1/2}$ and $e^{n+1/2}_{j+1/2}$ just computed.

15. Calculate $p^{n+1}_{j+1/2}$ from an equation of state using $e^{n+1}_{j+1/2}$ and $\rho^{n+1}_{j+1/2}$.

16. Advance time

$$\text{total time} = t^{n} + \Delta t^{n+1/2}.$$

17. Find a new Δt,

$$C^{n+1}_{j+1/2} = (\gamma_{j+1/2} p^{n+1}_{j+1/2} / \rho^{n+1}_{j+1/2})^{1/2},$$

$$\Delta t1^{n+1/2} = \operatorname*{Min}_{j}\left\{\frac{(x^{n+1}_{j+1} - x^{n+1}_{j})}{\text{Const } C^{n+1}_{j+1/2}}\right\},$$

where $C^{n+1}_{j+1/2}$ is the local sound speed in a zone, and the constant is approximately 2 (see Ref. [6]).

$$\Delta t2^{n+1/2} = \operatorname*{Min}_{j}\left\{\frac{\text{Const } V^{n+1/2}_{j+1/2}}{| V^{n+1}_{j+1/2} - V^{n}_{j+1/2} |}\right\}$$

where the constant is chosen such that no zone may appreciably change its volume during one cycle.

$$\Delta t3^{n+1/2} = \operatorname*{Min}_{j}\left\{\frac{\text{Const } m^{n+1/2}_{j+1/2}}{S^{n+1}_{j+1/2} \mid \dot{m}^{n+1}_{j+1/2} \mid}\right\},$$

where the constant is of the order of 10^{-2}. This condition is discussed in more detail in Sections III and IV. It insures that the mass of a zone does not change

by a large fraction during one cycle. When energy sources or sinks are large, similar Δt criteria are used for them.

$$\Delta t^{n+1/2} = \text{Min}(\Delta t1^{n+1/2}, \Delta t2^{n+1/2}, \Delta t3^{n+1/2})$$

$$\Delta t^{n} = \tfrac{1}{2}(\Delta t^{n-1/2} + \Delta t^{n+1/2})$$

$$\Delta t^{n+1} = \Delta t^{n+1/2}(\text{an estimate})$$

18. Advance cycle number

$$n + 1 \rightarrow n$$

go back to 1. for the next cycle.

III. Stability of the "Almost-Lagrangian" Differential Equations

In order to investigate the stability of the "almost-Lagrangian" differential equations, an approach similar to the one used by Von Neumann and Richtmyer for the traditional Lagrangian equations is considered [6]. The PUFL conservation equations, (4), (8), and (12), are restated below, with the following relationships used for:

specific internal energy, $e = \dfrac{p}{(\gamma - 1)\rho}$,

wall shear stress $\tau_w = \dfrac{1}{2} C_f \rho u^2$,

mass addition $\dfrac{\dot{m}S}{m} = \dfrac{1}{m}\dfrac{dm}{dt} = \beta$,

one-dimensional plane flow $\dfrac{1}{V}\dfrac{dV}{dt} = \dfrac{\partial u}{\partial x}$.

Continuity: $$\frac{d\rho}{dt} = -\rho \frac{\partial u}{\partial x} + \rho\beta. \tag{17}$$

Momentum: $$\frac{du}{dt} = \beta(u_w - u) - \frac{1}{\rho}\frac{\partial}{\partial x}(p + q) - \frac{C_f u^2}{r}. \tag{18}$$

Energy: $$\frac{1}{\rho(\gamma - 1)}\frac{dp}{dt} = \frac{p}{\rho^2(\gamma - 1)}\frac{d\rho}{dt} + \beta\left[\frac{(u - u_w)^2}{2} + e_w + \frac{p}{(\gamma - 1)\rho}\right] - \frac{(p + q)}{\rho}\frac{\partial u}{\partial x} + \frac{C_f u^2 \mid u \mid}{r} + \frac{\dot{H}}{m}. \tag{19}$$

Artificial viscosity: $$q = \rho(C_0 \Delta x)^2 \frac{\partial u}{\partial x}\left|\frac{\partial u}{\partial x}\right| \text{(Using } C_1 = 0). \tag{20}$$

For investigation of the effect of small perturbations $\delta u, \delta\rho, \delta p, \delta q$ which are imposed on a solution u, ρ, p, q, the solution variables in Eqs. (17) to (20) are replaced by $u \to u + \delta u$, $\rho \to \rho + \delta\rho$, $p \to p + \delta p$, $q \to q + \delta q$. The resulting equations of first variation, which neglect terms of order δ^2 are:

$$\text{Continuity:} \quad \left\{\frac{\partial}{\partial t} - \beta + \frac{\partial u}{\partial x}\right\}\delta\rho + \rho\frac{\partial}{\partial x}\delta u = 0, \tag{21}$$

$$\text{Momentum:} \quad \left\{\frac{\partial u}{\partial t} - (u_w - u)\beta + \frac{C_f u^2}{r}\right\}\delta\rho + \left\{\rho\frac{\partial}{\partial t} + \beta\rho + 2\rho\frac{|u|C_f}{r}\right\}\delta u + \frac{\partial}{\partial x}\delta p + \frac{\partial}{\partial x}\delta q = 0, \tag{22}$$

$$\text{Energy:} \quad \left\{\frac{\partial p}{\partial t} - p\frac{\partial}{\partial t} - 2\beta\rho(\gamma-1)\left[e_w + \frac{(u-u_w)^2}{2}\right] + p\beta - 2\rho(\gamma-1) \times \left(\frac{C_f|u|^3}{r} + \frac{\dot{H}}{m}\right) + (\gamma-1)(p+q)\frac{\partial u}{\partial x}\right\}\delta\rho + \left\{-\beta\rho^2(\gamma-1)(u-u_w) - \frac{C_f}{r}(\gamma-1)u^2\rho^2 3 + (\gamma-1)\rho(p+q)\frac{\partial}{\partial x}\right\}\delta u + \left\{\rho\frac{\partial}{\partial t} - \frac{\partial\rho}{\partial t} + \rho\beta + (\gamma-1)\rho\frac{\partial u}{\partial x}\right\}\delta p + \left\{(\gamma-1)\rho\frac{\partial u}{\partial x}\right\}\delta q = 0, \tag{23}$$

$$\text{Artificial Viscosity:} \quad \left\{(C_0\Delta x)^2\frac{\partial u^2}{\partial x}\right\}\delta\rho + \left\{2(C_0\Delta x)^2\rho\left|\frac{\partial u}{\partial x}\right|\frac{\partial}{\partial x}\right\}\delta u + \delta q = 0. \tag{24}$$

Equations (21)–(24) are a set of linear simultaneous equations for δu, δp, and δq, whose coefficients depend on the solutions, ρ, u, p, and q. For investigation of the effect of perturbations that are rapidly varying compared with ρ, u, and p, this mean solution may be thought of as being constant in a small region in the (x, t) plane denoted by 0.

In the following, only the region away from the shock front is considered, where $q \approx 0$, and the smoothly varying approximation for the solution ρ, u, p is frequently very good. In most physical situations, the significant region for ablation or condensation effects occurs at least a little distance behind the shock (i.e. behind the few zones where $q \neq 0$). Generally, for pipe flows, the aspect ratio (width/diameter) for the shock front itself is relatively small. Thus, mass perturbations introduced along the pipe walls are probably not handled realistically by a one-dimensional representation within the shock front itself.

Solutions having the form

$$\delta\rho = \delta\rho_0 e^{ikx+\alpha t}, \qquad \delta u = \delta u_0 e^{ikx+\alpha t}, \qquad \delta p = \delta p_0 e^{ikx+\alpha t} \tag{25}$$

are then sought, where $\delta\rho_0$, δu_0, δp_0, k, and α are constants and k is real. (For instance, the quantity $\delta\rho_0$ may be considered as a small perturbation on the density ρ_0 induced by a change in mass.) Substituting (25) into Eqs. (21) to (23) and rewriting Eq. (24) as $\delta q = 0$ results in three simultaneous homogeneous linear equations in $\delta\rho_0$, δu_0, and δp_0. When the determinant of these equations is set equal to zero, the following equation results:

$$
\begin{aligned}
&-k^2\rho\left\{\frac{\partial p}{\partial t} - p\alpha - 2\beta\rho(\gamma-1)[e_w + (u-u_w)^2/2]\right.\\
&\left.+ p\beta - 2\rho(\gamma-1)\left(\frac{C_f \mid u \mid^3}{r} + \frac{\dot{H}}{m}\right) + (\gamma-1)\,p\,\frac{\partial u}{\partial x}\right\}\\
&+\left(\alpha - \beta + \frac{\partial u}{\partial x}\right)\left(\rho\alpha + \rho\beta + 2\rho\,\frac{uC_f}{r}\right)\left(\rho\alpha + \rho\beta - \frac{\partial\rho}{\partial t} + (\gamma-1)\,\rho\,\frac{\partial u}{\partial x}\right)\\
&-ik\rho\left(\frac{\partial u}{\partial t} - u_w - u\beta + \frac{C_f u^2}{r}\right)\left(\rho\alpha - \frac{\partial\rho}{\partial t} + \rho\beta + (\gamma-1)\,\rho\,\frac{\partial u}{\partial x}\right)\\
&-ik\left(\alpha - \beta + \frac{\partial u}{\partial x}\right)\left(-\beta\rho^2(\gamma-1)(u-u_w) - \frac{C_f}{r}(\gamma-1)\,u^2\rho^2 3 + ik\rho(\gamma-1)\,p\right) = 0.
\end{aligned}
\tag{26}
$$

When only Fourier components with very large k are considered, and when α is considered to be large, as in Ref. 6, the terms with α^3, $k^2\alpha$, $\alpha^2 k$ are dominant. Equation (26) then reduces to

$$\alpha^3 + \alpha(\gamma k^2 p/\rho) = 0 \tag{27}$$

or $\alpha^2 = -k^2\gamma p/\rho$, and α is imaginary or zero. Thus, small disturbances propagate without growth or decay (Eq. 25), which is the same result as obtained for the traditional Lagrangian equations [6]. The mass addition terms do not, therefore, introduce first-order modifications into the Lagrangian equations in a region where shocks are unimportant. (If a similar analysis is performed for the region where shocks are important, $q \neq 0$, again the traditional Lagrangian results are obtained.)

The equations of variation, (21) to (24), can be examined to determine which terms lead to the dominant terms in Eq. (26). When this is done for the traditional Lagrangian equations, Eqs. (21) to (24) exhibit the character of a wave equation in regions where $q \approx 0$ and a diffusion equation in the regions were $q \neq 0$. Stability criteria for the wave and diffusion equations are then found to be sufficient for the Lagrangian difference equations [6]. Since the mass addition, friction, and energy source terms introduce no first-order modifications in the traditional equations, it seems reasonable to investigate the traditional Lagrangian time-step conditions for the stability of these equations. This is discussed in the next section.

For situations where k cannot be considered large or where mass or energy are

being rapidly modified (large β or $\dot{H}$), the αk, α^2, k^2 terms in (26) increase in importance. When terms in αk, α^2, k^2, are considered, Eq. (26) reduces to an equation for α in the form

$$\alpha^3 + \alpha^2 A_1 + \alpha(A_2 + iA_3) + (A_4 + iA_5) = 0, \tag{28}$$

where A_1, A_2, A_3, A_4, and A_5 are all real. An analysis of (28) is then appropriate in order to determine conditions on the A's such that the real parts of all solutions for α are zero or negative. When the above conditions exist, the additional dominant terms may give the Lagrangian equations a character different than the wave equation. In such a case, the usual stability criteria based on the wave equation

Lagrangian equations with energy sources, instabilities are known to arise. These instabilities occur when an "appreciable" change in the energy of a zone takes place during a time step [1]. However, through experience, calculations with energy sources are known to give realistic results when the energy added per cycle is small compared with the energy already present in a zone, i.e. when $[e^{-1}(de/dt)]\,\Delta t$ is small. If introduced perturbations tend to grow in an unbounded manner with time, the time interval Δt over which they are allowed to grow might be made "sufficiently" small that the perturbations remain bounded during this interval. Thus, if a "sufficiently" small time interval Δt is used, a stable situation is achieved. This is the approach commonly used in calculations with energy sources.

By analogy, a similar control on $[m^{-1}(dm/dt)]\,\Delta t = \beta \Delta t$ may be appropriate for mass sources. This is also investigated in the next section where an indication of an acceptable limit for $\beta \Delta t$ is obtained.

IV. Flows with Mass Sources and Sinks

In the following, two idealized problems are to be considered: one with mass sources (the "ablation" problem), the other with mass sinks (the "condensation" problem). Both problems consider a uniform bounded mass of gas, flowing without shocks, in a frictionless ($\tau_w = 0$), rigid, constant-radius ($S/V = 2/r$), pipe. Since the bounded mass of gas is uniform, gradients of the flow parameters in the direction of flow are zero ($\partial p/\partial x = \partial u/\partial x = 0$). For a Lagrangian element in a constant radius pipe, $V^{-1}(dV/dt) = \partial u/\partial x$; hence, $\partial u/\partial x = 0$ implies that $dV/dt = 0$.

With the previously mentioned uniform bounded-mass assumptions, the PUFL conservation equations, (4), (8), and (12), reduce to:

$$\text{Continuity:} \quad \frac{d\rho}{dt} = \frac{\dot{m}S}{V}, \tag{29}$$

$$\text{Momentum:} \quad \frac{du}{dt} = \frac{\dot{m}S}{V}(u_w - u), \tag{30}$$

$$\text{Energy:} \quad \frac{de}{dt} = \frac{1}{m}\left\{\dot{m}S\left[\frac{(u-u_w)^2}{2} + e_w - e\right] + \dot{H}\right\}. \tag{31}$$

If $\dot{m}S/m = m^{-1}(dm/dt) = \beta$, where β is considered to be a constant, then

$$dm/m = \beta\, dt \quad \text{and} \quad m = m_0 \exp(\beta t), \tag{32}$$

where m_0 denotes the mass at $t = 0$. In both the mass source ($\beta > 0$) and mass sink ($\beta < 0$) problems, the flow is considered to "interact" with the walls in such a way that the total energy of the flow remains constant. It is emphasized that the total energy requirement is used in the following only in order to easily obtain analytical solutions and is not required in PUFL where energy sources and sinks are permissible.

Mass Addition (the "Ablation" Problem)

In order to consider the mass addition problem we make the following additional assumptions. The entering mass is considered to have no velocity, $u_w = 0$, and a constant specific internal energy, $e_w = \text{constant}$. The rate of energy lost from the bounded mass, $-\dot{H}$, balances the rate of energy added by the entering mass, $\dot{m}Se_w$, so that the total energy of the bounded mass remains a constant. In this case, $\dot{m}$ and β are positive.

With the previously mentioned assumptions for mass addition, Eqs. (29) to (31) reduce to:

$$\text{Continuity:} \quad \frac{d\rho}{dt} = \beta\rho = \frac{\rho}{m}\frac{dm}{dt}; \tag{33}$$

$$\text{Momentum:} \quad \frac{du}{dt} = \beta u = -\frac{u}{m}\frac{dm}{dt}; \tag{34}$$

$$\text{Energy:} \quad \frac{de}{dt} = \beta\left\{\frac{u^2}{2} - e\right\} = \frac{1}{m}\left\{\frac{u^2}{2} - e\right\}\frac{dm}{dt}. \tag{35}$$

The conservation equations (33) to (35) may be readily integrated. The results are shown below.

$$\frac{\rho}{\rho_0} = \frac{m}{m_0}, \tag{36}$$

$$\frac{u}{u_0} = \frac{m_0}{m}, \tag{37}$$

$$\frac{e}{e_0} = \frac{m_0}{m}\left[1 + \frac{u_0^2}{2e_0}\left(1 - \frac{m_0}{m}\right)\right], \tag{38}$$

where ρ_0, u_0, e_0 denote values at $t = 0$ when $m = m_0$ and $\rho_0 V = m$ (note, $V = V_0$). The above expression for e/e_0 vs m/m_0 with various values of $K = u_0^2/2e_0$, is shown in Fig. 4.

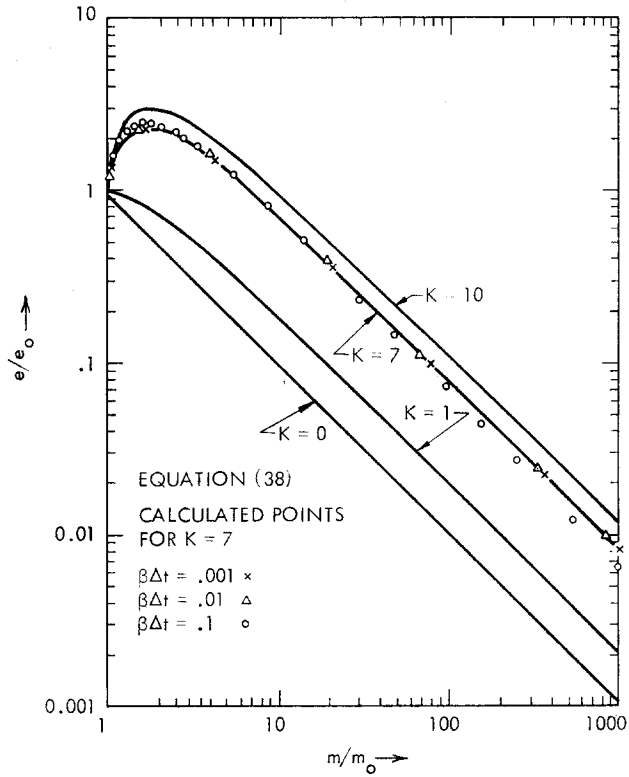

FIG. 4. Specific energy vs mass for a uniform flow with mass addition.

Some PUFL calculations are now considered for the idealized "ablating" flow just discussed. The mass flux entering the flow is calculated for each zone every cycle as:

$$\dot{m}_{j+1/2}^{n+1/2} = \frac{\beta m_{j+1/2}^{n+1/2}}{S_{j+1/2}^{n+1/2}}, \tag{39}$$

where β is a positive input constant. Hence, each cycle the ratio of "change in mass/mass" is a constant.

The first three calculations considered here are identical except for the quantity $\beta\,\Delta t$. In order to treat $\beta\,\Delta t$ as a constant, Δt is kept at a constant value throughout

each calculation. Also, the same constant Δt is used in the three calculations. The three calculations have the following values for $\beta \Delta t$: $\beta \Delta t = +0.001, +0.01, +0.1$.

The constant Δt is chosen such that it satisfies $\Delta t = \Delta x / 2C_{\max}$, where Δx remains the same (since $\partial u/\partial x = 0$) and $C_{\max}$ is the maximum sound speed encountered during the calculation. The identical initial conditions use $K = u_0^2/2e_0 = 7$. The calculations are allowed to run until the mass per zone increases to 1000 times its initial value. The results of these three calculations are also shown in Fig. 4.

Mass Deletion (the "Condensation" Problem)

In order to consider mass deletion or the condensation problem, we make slightly different assumptions about u_w and e_w. The exiting mass is originally a part of the flow, hence it has the same energy as the flow and leaves with $u_w = u$ and $e_w = e$. In this idealized condensation process, the exiting material is assumed to give up all of its energy when leaving, and the energy is assumed to all come back into the flow as an energy source. Therefore, $\dot{H} = \dot{m}S(e_w + u_w^2/2)$ where $\dot{H}$ is positive, and $\dot{m}$ is negative for mass sinks. Hence, as in the mass addition case, the total energy of the flow remains constant and the mass which becomes "wall material" carries no energy with it.

With these assumptions, Eqs. (29) to (31) reduce to:

$$\text{Continuity:} \qquad \frac{d\rho}{dt} = \beta\rho = \frac{\rho}{m}\frac{dm}{dt}\,; \tag{40}$$

$$\text{Momentum:} \qquad du/dt = 0; \tag{41}$$

$$\text{Energy:} \qquad \frac{de}{dt} = \dot{H} = \frac{\dot{m}S}{m}\left(e + \frac{u^2}{2}\right) = -\frac{1}{m}\left(e + \frac{u^2}{2}\right)\frac{dm}{dt}\,. \tag{42}$$

These equations may be readily integrated to obtain:

$$\text{Continuity:} \qquad \frac{\rho}{\rho_0} = \frac{m}{m_0}\,; \tag{43}$$

$$\text{Momentum:} \qquad u/u_0 = 1; \tag{44}$$

$$\text{Energy:} \qquad \frac{e}{e_0} = \frac{m_0}{m}\left(1 + \frac{u_0^2}{2e_0}\right) - \frac{u_0^2}{2e_0}\,. \tag{45}$$

Figure 5 shows e/e_0 vs m/m_0 from Eq. (45) for various values of $K = u_0^2/2e_0$.

Three PUFL calculations are now considered for the idealized "condensing" flow just discussed. These calculations are like those for the "ablating" flow, except for the previously described energy prescriptions required to conserve

total energy. The initial conditions are again identical and have $K = u_0^2/2e_0 = 7$. For these calculations, β is negative; and mass is being depleted by a flux which is also calculated by Eq. (39). Again a constant Δt is chosen such that it satisfies

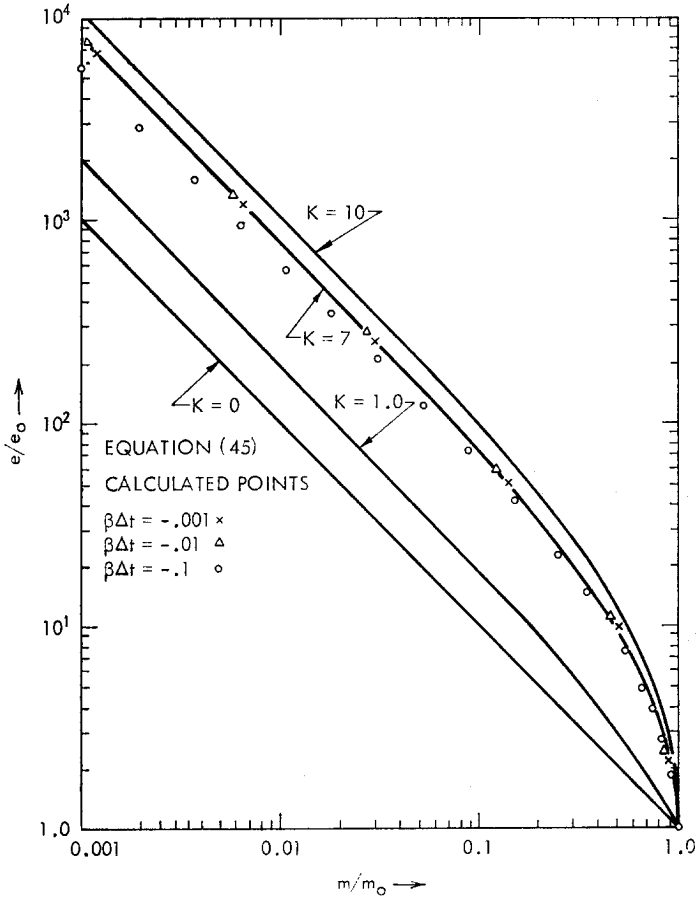

FIG. 5. Specific energy vs mass for a uniform flow with mass deletion.

$\Delta t = \Delta x/2C_{\max}$. The problems are run until the mass per zone is 1/1000 of its original value. Notice, from Fig. 5, that the specific energy increases by a factor of $\sim 10^4$ when the mass decrease by a factor of 10^3 for $K = 7$. Hence, the sound speed increases a factor of $\sim 10^2$ during the calculation, $[C = (\gamma p/\rho)^{1/2} = (\gamma(\gamma - 1)e)^{1/2}]$. The constant Δt used in these calculations is therefore based on a sound speed 10^2 larger than its initial value, $C_{\max} \sim 10^2 C_0$.

The results of three calculations using the same Δt but varying β such that $\beta\,\Delta t = -0.001$, $-.01$, and -0.1 are also shown in Fig. 5.

From Figs. 4 and 5 it can be seen that the calculations deviate from the analytical solution for values of $|\beta\,\Delta t|$ of about 0.1. However, for values of $|\beta\,\Delta t|$ of the order of 0.01, the agreement with the analytical solution is excellent. It thus appears that in order to obtain realistic and stable solutions, the mass change per time step should be of the order of 1/100 of the mass currently present in the zone, $\beta\,\Delta t = O(0.01)$.

To demonstrate that a time-step condition of the form

$$\Delta t \leqslant \frac{\Delta x}{2c} = \frac{\Delta x}{2[\gamma(\gamma - 1)\,e]^{1/2}} \tag{46}$$

is at least necessary throughout a calculation, even when $|\beta\,\Delta t| = 0.001$, we consider another set of calculations. These calculations again simulate the idealized "condensation" case with identical initial conditions of $K = u_0^2/2e_0 = 7$. Both Δt and β remain constant and $\beta\,\Delta t = -0.001$ throughout each calculation. The zone sizes, Δx, initially are, and remain, the same in all of the calculations. The calculations differ by their value for Δt (and hence β). The Δt's used by the various calculations each use a different value for e in (46). Table I outlines: (1) the calculation considered, (2) the value of e used in (46) to define the constant Δt for that calculation, (3) an approximate value of e in the calculation when it becomes unstable and completely degenerates.

TABLE I

CALCULATIONS FOR DETERMINING A SUFFICIENT Δt

Calculation No.	e used for Δt in Eq. (46)	e by which calculation degenerates
1	e_0	$e \sim 4e_0$
2	$4e_0$	$e \sim 16e_0$
3	$25e_0$	$e \sim 100e_0$
4	$100e_0$	$e \sim 400e_0$
5	$400e_0$	$e \sim 1600e_0$
6	$2{,}500e_0$	[a]
7	$10{,}000e_0$	[a]

[a] Calculation is stable at $e/e_0 = 8000$, where it was terminated.

Calculation No. 7 listed in Table I is the one shown in Fig. 5 for $\beta\,\Delta t = -0.001$. Calculations 6 and 7 remain stable throughout the calculation and give excellent agreement with the analytical solution. Before Calculations 1–5 degenerate, they also show excellent agreement with the stable calculations.

The last column in Table I indicates that a time-step control of the form of (46) is required. It also indicates that the Δt used in Calculations 1–5 is sufficient for stability until the energy increases by a factor of approximately 4. Thus, in Eq. (46), instead of using e, it is sufficient to use $e/4$. This indicates that the constant required in (46) may be closer to 1.0 than 2.0. A constant of 1.0 compares with the results for the traditional Lagrangian equations in a region where shocks are unimportant [6].

Additional test calculations have been considered with a constant or variable $\dot{m}$, hence a variable β. As indicated by the previous results, these calculations have remained stable and agreed with the analytical solution when Δt was limited by

$$\Delta t \leqslant \frac{\text{Const } m_{j+1/2}^{n+1/2}}{\dot{m}_{j+1/2}^{n+1/2} S_{j+1/2}^{n+1/2}}, \tag{47}$$

where the constant is of the order of 10^{-2}.

For most physical situations which involve mass sources or sinks, Eq. (47) does not appear to present a significant limitation. Hence, it is felt that the "almost-Lagrangian" approach is feasible.

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APPENDIX A. Derivation of Surface Forces Acting on an Element

Consider an element of total surface area S_T and volume V (Fig. 6). In this figure, $\hat{n}$ is the unit outward normal vector at an arbitrary point P on the surface. At P, the stress tensor $\boldsymbol{\sigma}$ operates on the surface normal vector $\hat{n}$ to produce the stress vector $\mathbf{s}$. Here $\mathbf{s}$ is the force per unit area exerted *on* the surface of the element *by* the medium into which $\hat{n}$ is pointing. If $\boldsymbol{\sigma}$ is defined at all points P on the surface, then the total surface forces acting on the element are

$$\mathbf{F} = \int_{S_T} \hat{n}\boldsymbol{\sigma}\, dS_T\,. \tag{A1}$$

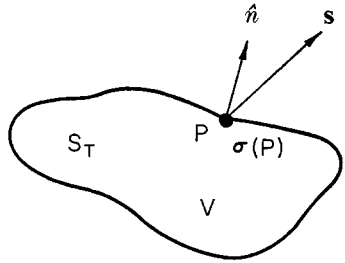

FIG. 6. A bounded mass of gas with total surface area S_T and volume V; $\hat{n}$ is a unit outward normal vector at an arbitrary point P on the surface. At P, the stress tensor $\boldsymbol{\sigma}$ operates on $\hat{n}$ to produce $\mathbf{s}$.

The divergence of a tensor $\boldsymbol{\sigma}$ may be defined [8] as

$$\operatorname{div} \boldsymbol{\sigma} = \operatorname*{Lim}_{V\to 0} \frac{1}{V} \int_{S_T} \hat{n}\boldsymbol{\sigma}\, dS_T\,. \tag{A2}$$

Assuming that the above limit exists for an element, the total surface forces [Eq. (A1)] may be written

$$\mathbf{F} = V \operatorname{div} \boldsymbol{\sigma} = \int_{S_T} \hat{n}\boldsymbol{\sigma}\, dS_T \tag{A3}$$

The stress tensor $\boldsymbol{\sigma}$ may be defined as

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau}, \tag{A4}$$

where $-p$ is the thermodynamic pressure, $\mathbf{I}$ is the idemfactor, and $\boldsymbol{\tau}$ is the stress deviator. Then, Eq. (A3) becomes

$$\mathbf{F} = V \operatorname{div} \boldsymbol{\sigma} = -\int_{S_T} \hat{n}p\mathbf{I}\, dS_T + \int_{S_T} \hat{n}\boldsymbol{\tau}\, dS_T = -V \operatorname{div} p\mathbf{I} + V \operatorname{div} \boldsymbol{\tau}. \tag{A5}$$

Consider for the present only the first term in Eq. (A5).

$$-\int_{S_T} \hat{n}p\mathbf{I}\, dS_T = -V \operatorname{div} p\mathbf{I}.$$

In general,

$$-V \operatorname{div} p\mathbf{I} = -V\left\{\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}, \frac{\partial p}{\partial x_3}\right\}.$$

But, for one-dimensional flow, the above is simply $-V\partial p/\partial x_1$ since

$$\partial p/\partial x_2 = \partial p/\partial x_3 = 0.$$

Hence Eq. (A5) may now be written as

$$\mathbf{F} = V \operatorname{div} \boldsymbol{\sigma} = -V\frac{\partial p}{\partial x} + \int_{S_T} \hat{n}\boldsymbol{\tau}\, dS_T. \tag{A6}$$

For the description of the flow of gas through a pipe, the internal viscous stresses of the gas are usually negligible compared with the stress exerted by the wall on the gas. This stress exerted by the wall on the gas acts as a retarding force on the flow and may be represented as $-\tau_w S$, where τ_w is the shearing stress that acts over a surface of area S; τ_w is related to C_f, the dimensionless coefficient of skin friction [8], [9] by

$$\tau_w = \tfrac{1}{2}C_f \rho u^2. \tag{A7}$$

If the integral in Eq. (A6) is approximated by $-\tau_w S$, the total surface forces acting on an element may be written as

$$\mathbf{F} = V \operatorname{div} \boldsymbol{\sigma} = -V(\partial p/\partial x) - \tau_w S. \tag{A8}$$

APPENDIX B. Derivation of the Rate of Work Done on an Element

The rate of work done *by* an element against surface forces may be represented by

$$\int_{S_T} (\hat{n}\boldsymbol{\sigma}) \cdot \mathbf{u}\, dS_T. \tag{B1}$$

The quantity $\hat{n}\boldsymbol{\sigma} \cdot u$ may be rewritten as

$$\hat{n}\boldsymbol{\sigma} \cdot \mathbf{u} = \mathbf{u} \cdot \hat{n}\boldsymbol{\sigma} = \hat{n} \cdot (\mathbf{u}\boldsymbol{\sigma}^*) = \hat{n} \cdot \mathbf{u}\boldsymbol{\sigma}, \tag{B2}$$

where σ^* is the transpose of σ, and $\sigma^* = \sigma$, since the stress tensor is symmetric. For a vector α,

$$\operatorname{div} \alpha = \lim_{V \to 0} \frac{1}{V} \int_{S_T} \hat{n} \cdot \alpha \, dS_T . \tag{B3}$$

Assuming that the above limit exists and using Eqs. (B3) and (B2) in Eq. (B1), the result is

$$\int_{S_T} \hat{n}\sigma \cdot \mathbf{u} \, dS_T = V \operatorname{div} \mathbf{u}\sigma. \tag{B4}$$

The following identity for any vector $\mathbf{a}$ and tensor $\mathbf{b}$ is useful.

$$\operatorname{div} \mathbf{ab} = \mathbf{a} \cdot \operatorname{div} \mathbf{b} + \mathbf{b} \cdot \cdot \operatorname{grad} \mathbf{a}.$$

When this identity is used in Eq. (B4), one obtains

$$V \operatorname{div} \mathbf{u}\sigma = V\{\mathbf{u} \cdot \operatorname{div} \sigma + \sigma \cdot \cdot \operatorname{grad} \mathbf{u}\}. \tag{B5}$$

Since $\sigma = -p\mathbf{I} + \tau$,

$$V\sigma \cdot \cdot \operatorname{grad} \mathbf{u} = V(-p\mathbf{I} \cdot \cdot \operatorname{grad} \mathbf{u} + \tau \cdot \cdot \operatorname{grad} \mathbf{u}). \tag{B6}$$

Considering the first part of the above, in one dimension,

$$-p(\mathbf{I} \cdot \cdot \operatorname{grad} \mathbf{u}) = -p(\partial u/\partial x) = -p\left(\frac{1}{V}\frac{dV}{dt}\right); \tag{B7}$$

hence Eq. (B6) becomes

$$V\sigma \cdot \cdot \operatorname{grad} \mathbf{u} = -p(dV/dt) + V\tau \cdot \cdot \operatorname{grad} \mathbf{u}. \tag{B8}$$

The term $V\tau \cdot \cdot \operatorname{grad} \mathbf{u}$ is the irreversible dissipative work done *on* the element *by* surface stresses. In one-dimensional gas flow through a pipe, the pipe walls exert stress on the element, which results in irreversible frictional heating of the element. The term $V\tau \cdot \cdot \operatorname{grad} \mathbf{u}$ may be expressed in one-dimensional pipe flow as $\tau_w S \mid u \mid$. Hence Eq. (B4) may be written

$$\int_{S_T} \hat{n}\sigma \cdot \mathbf{u} \, dS_T = V\mathbf{u} \cdot \operatorname{div} \sigma - p(dV/dt) + \tau_w S \mid u \mid. \tag{B9}$$

APPENDIX C. Expressions Describing the Ablative Process in Turbulent Flow

An expression for heat transfer of the type commonly used in turbulent flow is discussed.

The heat flux out of a zone may be written as

$$q = \alpha(T - T_w), \tag{C1}$$

where T_w is the temperature of the wall, T is the temperature of the gas flow in the pipe, and α is the local coefficient of heat transfer. For $T \gg T_w$ this becomes

$$q = \alpha T. \tag{C2}$$

By combining four dimensionless numbers,

$N_u = \alpha R/k$ Nusselt number,
$R_e = (\rho u R)/\mu$ Reynolds number, where μ = viscosity,
$P_R = (C_p \mu)/k$ Prandtl number, where C_p is the specific heat at constant pressure,
$ST = N_u/(R_e P_R) = \alpha/(\rho C_p u)$ Stanton number, sometimes denoted by C_H, the dimensionless coefficient of heat transfer.

The expression $C_H = ST = Nu/R_e P_R$ for the dimensionless coefficient of heat transfer is derived by Schlichting [9]. The assumption is made in this derivation that both the main flow turbulent and sublayer laminar Prandtl numbers are equal to 1.0. This implies the physical assumption that the same mechanism causes the exchange of momentum and of heat. This physical assumption is frequently referred to as the Reynolds analogy, which may be written $C_H = \frac{1}{2} C_f$. An expression for α can be obtained and used in the heat-flux term as follows.

$$q = [N_u/(R_e P_R)]\, \rho u C_p T = C_H \rho u C_p T. \tag{C3}$$

Using ideal gas relationships, $e = C_v T$, and $\gamma = C_p/C_v$,

$$C_p T = (C_p e)/C_v = (\gamma P)/[(\gamma - 1)\, \rho], \tag{C4}$$

where C_v is specific heat at constant volume, the heat flux term may be written

$$q = C_H(\gamma P u)/(\gamma - 1). \tag{C5}$$

The above approach for obtaining the heat flux is certainly only an approximation. However, this simplified approach may adequately describe the general physical effects for a number of turbulent-flow problems.

The specific heat of ablation is the energy require to ablate a unit mass of pipe material and may be approximated by [10]

$$q^* = E_v + \eta h, \tag{C6}$$

where the quantities for the pipe material are

$E_v = C_p \Delta T + h_f + h_v$(an input constant),
$C_p \Delta T$ = enthalpy required to raise a unit mass of wall material to vaporization temperature,
h_f = specific heat of fusion,
h_v = specific heat of vaporization,
h = specific total enthalpy of the gas flow,
η = turbulent transpiration coefficient.

The coefficient η is an approximation for the energy loss to the boundary layer and surrounding media [10]. It may be treated as an input constant which depends on the material of the pipe. The specific total enthalpy of the gas flow is calculated as

$$h = e + (p/\rho) + \tfrac{1}{2}u^2 = \gamma e + \tfrac{1}{2}u^2. \tag{C7}$$

The rate of heat supplied *to* a zone in the PUFL equations has been denoted by $\dot{H}_{j+1/2}$. In order to simulate the turbulent ablation process, the heat lost from a zone may be calculated by

$$\dot{H}_{j+1/2} = -q_{j+1/2}S_{j+1/2}.$$

The mass flux is then calculated as

$$\dot{m}_{j+1/2} = \frac{q_{j+1/2}}{q^*_{j+1/2}} = \frac{\dfrac{\text{energy}}{\text{area–time}}}{\dfrac{\text{energy}}{\text{mass}}} = \frac{\text{mass}}{\text{area–time}}. \tag{C8}$$

APPENDIX D. NOTATION

A	cross-sectional area normal to flow in pipe
C	local sound speed
C_f	dimensionless friction coefficient for tubes
C_H	dimensionless coefficient of heat transfer
C_p	specific heat at constant pressure
C_v	specific heat at constant volume
C_0^2, C_1	input constants used in the calculation of Q (typical values are $C_0^2 \sim 3$, $C_1 \sim 1$)
e	specific internal energy of material in a zone
e_w	specific internal energy of mass entering a zone
E_v	$= h_f + h_v + C_p\Delta T$, an input constant to PUFL
$\mathbf{F}$	body and surface forces acting on a zone
h	specific total enthalpy
h_f	specific heat of fusion
h_v	specific heat of vaporization
$\dot{H}$	rate of heat supplied to an element
i	$(-1)^{1/2}$
$\mathbf{I}$	idemfactor
k	thermal conductivity
m	mass

$\dot{m}$	mass flux (mass/area–time)
$\hat{n}$	unit outward normal vector
N_u	Nusselt number
p	pressure
P_R	Prandtl number
q	heat flux out of a zone (energy/area–time)
q^*	specific heat of ablation
Q	artificial dissipative term added to pressure to simplify numerical integration (see ref. 4)
R	radius of pipe
R_e	Reynold's number
$\mathbf{s}$	stress vector
S	surface area of zone touching the pipe wall
S_T	total surface area of a zone
ST	Stanton number
t	time
T	temperature of gas flow in pipe
T_w	temperature of wall
$\mathbf{u}$	vector velocity
u	scalar velocity [in one-dimension $\mathbf{u} = (u)$]
$\lvert u \rvert$	absolute value of the velocity
u_w	velocity of mass entering a zone
V	volume of a zone
x	distance, measured along the center line of the pipe
α	local coefficient of heat transfer
β	$= \frac{1}{m}\frac{dm}{dt} = \frac{\dot{m}S}{m}$
γ	$= C_p/C_v$, ratio of specific heats
Δt	an interval of time
ΔT	temperature rise required to vaporize wall material
η	turbulent transpiration coefficient
μ	viscosity
ρ	density
$\boldsymbol{\sigma}$	stress tensor
τ_w	shearing stress at the wall
$\boldsymbol{\tau}$	stress deviator
$\cdot\cdot$	a tensor product resulting in a tensor

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